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Solutions of Extended Supersymmetric Matrix Models for Arbitrary Gauge Groups

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Abstract

Energy eigenstates for $N = 2$ supersymmetric gauged quantum mechanics are found for the gauges groups $SU(n)$ and $U(n)$. The analysis is aided by the existence of an infinite number of conserved operators. The spectum is continuous. Perturbative eigenstates for $N > 2$ are also presented, a case which is relevant for the conjectured description of M theory in the infinite momentum frame.

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I. Introduction

A view has emerged that different superstring theories are various limits of one unifying theory. That theory is M theory, an eleven-dimensional system whose low-energy limit is $D = 11$ supergravity and whose compactifications to ten dimensions on a circle and an interval yield type IIA and $E_8 \times E_8$ heterotic superstrings.[1]-[6] Dualities [6]-[16], as well as new string degrees of freedom such as D branes [17, 18] and supermembranes [19, 20, 21], have played a role in obtaining this unification. Supersymmetric matrix quantum mechanics has been useful in gaining insight into D-branes, supermembranes and M theory.[22]-[25] Although a precise *covariant* formulation of M theory is still lacking, it has been conjectured that the $n \to \infty$ limit of an $SU(n)$-matrix quantum mechanics system with $N = 16$ supersymmetry describes M theory in the infinite momentum frame.[25] Hence, any progress in understanding such systems is of interest.

Ground states of supersymmetry quantum mechanics are often of the from $\exp(\pm W)$, where W is the superpotential.[26, 27]. Excited energy states are not known with one exception: All such states of the $N = 2$ supersymmetric $SU(2)$ gauge theory have been found.[27] In this letter, we obtain energy eigenstates for the $U(n)$ and $SU(n)$ systems for general n. Given the interest in the $SU(\infty)$ case, out results should be of use in future research. The $N = 16$ case, which is relevant for M theory is not solved. However, perturbative $N = 16$ energy eigenstates, for which the coupling constant is set to zero, are found; this is non-trivial due to the requirement of satisfying the Gauss law constraints. Our results are a first step toward a perturbative analysis of the M theory proposal of ref.[25].

II. Particular Solutions for the $N = 2$ Case

The $N = 2$ quantum-mechanic gauge theory involves a real gauge potential A_B, a real scalar ϕ_B, and a complex fermion ψ_B, all in the adjoint representation. Here, B, which is a gauge index, runs over the number n_G of generators of the Lie group G, e.g. $n_G = n^2 - 1$ for $SU(n)$ and $n_G = n^2$ for $U(n)$. The lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2} \left(\mathcal{D}_t \phi\right)_A \left(\mathcal{D}_t \phi\right)_A + i\bar{\psi}_A \left(\mathcal{D}_t \psi\right)_A - igf_{ABC}\bar{\psi}_A \phi_B \psi_C \quad , \tag{1}$$

where the covariant derivative $\mathcal{D}_t$ on any field φ is $\left(\mathcal{D}_t \varphi\right)_A \equiv \partial_t \varphi_A - gf_{ABC} A_B \varphi_C$

and where g is the gauge coupling. Here and elsewhere, the presence of a repeated index indicates summation. The f_{ABC} are structure constants: $[\sigma_A, \sigma_B] = i f_{ABC}\sigma_C$, where σ_A are the generators of the Lie algebra of G. We choose the σ_A to be matrices satisfying

$$Tr\,(\sigma_A \sigma_B) = \delta_{AB} \quad . \tag{2}$$

It is straightforward to quantize the gauge system governed by Eq.(1). The hamiltonian is

$$H = \frac{1}{2}\pi_A\pi_A + ig f_{ABC}\bar{\psi}_A\phi_B\psi_C \quad , \tag{3}$$

where ϕ_A and π_B, as well as ψ_A and $\bar{\psi}_B$ are conjugate variables satisfying $[\phi_A, \pi_B] = i\delta_{AB}$ and $\left\{\psi_A, \bar{\psi}_B\right\} = \delta_{AB}$. States $|\, s\rangle$ must satisfy the Gauss law constraints

$$G_A|\, s\rangle = 0 \quad , \tag{4}$$

where

$$G_A = f_{ABC}\left(\phi_B\pi_C - i\bar{\psi}_B\psi_C\right) \quad . \tag{5}$$

In other words, states must be gauge invariant. The degrees of freedom A_B do not enter the hamiltonian – their role has been replaced by these Gauss law constraints.

Note that

$$H = \frac{1}{2}\pi_A\pi_A + g\phi_A G_A \quad , \tag{6}$$

so that, on gauge-invariant states, H reduces to $\frac{1}{2}\pi_A\pi_A$. If it were not for Eq.(4), the theory would be free. One needs only to find the gauge-invariant plane wave states $|\, s\rangle$:

$$\frac{1}{2}\pi_A\pi_A|\, s\rangle = E_s|\, s\rangle \;\; , \qquad G_A|\, s\rangle = 0 \quad . \tag{7}$$

The lagrangian in Eq.(1) is invariant under the supersymmetry transformations generated by $Q = \psi_A\pi_A$ and $\bar{Q} = \bar{\psi}_A\pi_A$. As usual, the anticommutator of Q and $\bar{Q}$ yields the hamiltonian up to gauge transformations: $\left\{Q, \bar{Q}\right\} = \pi_A\pi_A = 2H - 2g\phi_A G_A$.

States are classified according to their fermion number. In particular, one can define the fermion vacuum $|\, +\rangle$ to be annihilated by all the ψ_A:

$$\psi_A|\, +\rangle = 0 \quad . \tag{8}$$

Assign fermion number 0 to $|+\rangle$. All other fermionic sectors are obtained by repeatedly applying $\bar{\psi}_A$. Since there are n_G such fermions, there are states with fermion number 0, 1, …, n_G, and the total number of fermionic Fock-space states is 2^{n_G}.

The case of $G = SU(2)$, for which $n_G = 3$, has been solved by M. Claudson and M. Halpern [27]. They found

$$
\begin{aligned}
|\,k\rangle_0 &= \frac{\sin(kr)}{kr}|+\rangle \quad , \\
|\,k\rangle_1 &= \frac{Q}{k}|\,k\rangle_0 = \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right]\frac{1}{r}\phi_A\psi_A|+\rangle \quad , \\
|\,k\rangle_2 &= \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right]\frac{1}{2r}\varepsilon_{ABC}\phi_A\bar{\psi}_B\bar{\psi}_C|+\rangle \quad , \\
|\,k\rangle_3 &= \frac{Q}{k}|\,k\rangle_2 = \frac{\sin(kr)}{kr}\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3|+\rangle \quad ,
\end{aligned} \tag{9}
$$

where the subscript p on $|\,k\rangle_p$ indicates the fermion number. Here, $r = \sqrt{\phi_A\phi_A}$, k is any non-negative real number, and ε_{ABC} is the completely antisymmetric tensor on three indices. The states in Eq.(9) are only plane-wave normalizable, as expected, since the spectrum is continuous.

The goal of this section is to obtain zero-fermion-number solutions for $G = U(n)$ and $G = SU(n)$. We first treat the $U(n)$ case. Let σ_A be the matrix generators in the fundamental representation, with a normalization respecting Eq.(2). Let $\Phi = \phi_A\sigma_A$ be an $n \times n$ matrix of scalar fields. Since a gauge-invariant functional of the ϕ_A depends only on the eigenvalues of Φ, write

$$
\Phi = \phi_A\sigma_A = U^{-1}DU \quad , \tag{10}
$$

where D is a diagonal matrix of the eigenvalues λ_j of Φ:

$$
D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad , \tag{11}
$$

and U is some unitary transformation. Let us look for solutions of the form $f(\lambda_1, \ldots, \lambda_n)\,|+\rangle$. Acting on such a state,

$$
\pi_A\pi_A \to -\frac{1}{\mathcal{M}^2}\sum_{j=1}^{n}\frac{\partial}{\partial\lambda_j}\mathcal{M}^2\frac{\partial}{\partial\lambda_j} \quad , \tag{12}
$$

where $\mathcal{M}^2$, the Vandermonde determinant measure factor, is the square of

$$\mathcal{M} = \prod_{1\leq i<j\leq n} (\lambda_i - \lambda_j) \quad . \tag{13}$$

After some guesswork, we have found solutions to $\frac{1}{2}\pi_A\pi_A f = Ef$. They are given by

$$f = \mathcal{M}^{-1} \exp\left[i\sum_{j=1}^{n} k_j\lambda_j\right] \quad , \tag{14}$$

where the "momenta" k_j are arbitrary real numbers. The energy eigenvalue is

$$E = \frac{1}{2}\sum_{j=1}^{n} k_j^2 \quad . \tag{15}$$

In verifying $\frac{1}{2}\pi_A\pi_A f = Ef$, one needs to used the identity

$$\sum_{\substack{k\neq j\\ k\neq i\\ j\neq i}} \frac{1}{\lambda_i - \lambda_j}\frac{1}{\lambda_i - \lambda_k} = 0 \quad , \tag{16}$$

which, incidently, arises in obtaining covariant superstring amplitudes for fermionic scattering processes [28].

The solutions in Eq.(14) behave badly when any two eigenvalues approach each other. It is possible, however, to obtain regular solutions by taking linear combinations of Eq.(14). A unique class of regular solutions is achieved by antisymmetrization using the permutations σ of the permutation group S_n on n elements:

$$\mid k\rangle_0 = \mathcal{N}\mathcal{M}^{-1}\sum_{\sigma\in S_n} (-1)^\sigma \exp\left[i\sum_j k_{\sigma(j)}\lambda_j\right] \mid +\rangle \quad , \tag{17}$$

where $\mathcal{N}$ is a normalization factor and $(-1)^\sigma$ is $+1$ for even permutations and -1 for odd permutations. It is easy to verfiy that $\mid k\rangle_0$ is non-singular as $\lambda_j \to \lambda_i$.

When $G = SU(n)$, a solution is obtainable from the $G = U(n)$ case because the system is separable. Select the generator index for the diagonal $U(1)$ subgroup to be the last one, n^2. For convenience, relabel this index as 0. Hence, $\sigma_{n^2} = \sigma_0$ where $\sigma_0 = I_n/\sqrt{n}$ and I_n is the $n\times n$ identity matrix. Since the laplacian on $U(n)$, as well as the hamiltonian, splits into a $U(1)$ part and an $SU(n)$ part as

$$\frac{1}{2}\pi_A\pi_A = \frac{1}{2}\pi_0\pi_0 + \frac{1}{2}\sum_{A=1}^{n^2-1}\pi_A\pi_A = H_{U(1)} + H_{SU(n)} - \phi_A G_A \quad , \tag{18}$$

solutions factorize into a product of a $U(1)$ wave function $f_{U(1)}$ times an $SU(n)$ wave function $f_{SU(n)}$ via

$$f_{U(n)} = f_{U(1)} f_{SU(n)} \quad . \tag{19}$$

Since $f_{U(1)}$ is a function of the sum of the eigenvalues and $f_{SU(n)}$ is a function of differences of eigenvalues, write

$$\lambda_j = (\lambda_j - \Sigma_\lambda) + \Sigma_\lambda \ , \qquad \text{where} \quad \Sigma_\lambda = \frac{1}{n} \sum_{j=1}^{n} \lambda_j \quad . \tag{20}$$

Performing the factorization in Eq.(19), one finds

$$f_{U(1)} = \exp\left[i n \Sigma_k \Sigma_\lambda \right] \ ,$$

$$f_{SU(n)} = \mathcal{N} \mathcal{M}^{-1} \sum_{\sigma \in S_n} (-1)^\sigma \exp\left[i \sum_{j=1}^{n} k_{\sigma(j)} \left(\lambda_j - \Sigma_\lambda \right) \right] \quad . \tag{21}$$

In Eq.(21), $k_{\sigma(j)}$ can be replaced by $k_{\sigma(j)} - \Sigma_k$, where

$$\Sigma_k = \frac{1}{n} \sum_{j=1}^{n} k_j \quad , \tag{22}$$

due to $\sum_{j=1}^{n} (\lambda_j - \Sigma_\lambda) = 0$. Hence, the $SU(n)$ wave functions really depend on only $n-1$ momenta since $\sum_{j=1}^{n} (k_j - \Sigma_k) = 0$. The energy separates into a $U(1)$-part $E_{U(1)}$ and an $SU(n)$-part $E_{SU(n)}$:

$$E = \frac{1}{2} \sum_{j=1}^{n} k_j^2 = \frac{1}{2} \bar{k}_0^2 + \frac{1}{2} \frac{n-1}{n} \sum_{j=1}^{n} \bar{k}_j^2 = E_{U(1)} + E_{SU(n)} \quad , \tag{23}$$

where the barred momenta are

$$\bar{k}_0 \equiv \sqrt{n} \Sigma_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} k_j \ , \qquad \bar{k}_j = \sqrt{\frac{n}{n-1}} \left(k_j - \Sigma_k \right) \quad . \tag{24}$$

The $(n-1)/n$ in $E_{SU(n)}$ compensates for the one constraint on the $\bar{k}_j$ of $\sum_{j=1}^{n} \bar{k}_j = 0$.

III. The Construction of Other Solutions

It turns out that the $N = 2$ system has infinitely many operators O that are conserved up to Gauss's law. These operators can generate new solutions from old ones. Let O be gauge invariant and not a functional of the ϕ_A. Then if $\mid s \rangle$ satifies

$H|\,s\rangle = E_s|\,s\rangle$, for H in Eq.(6) and if $O|\,s\rangle = |\,s'\rangle \neq 0$, then $|\,s'\rangle$ is also a gauge-invariant eigenstate of H with the same energy: $H|\,s'\rangle = E_s|\,s'\rangle$. The proof is straightforward. Constructing O satisfying these criteria is simple. It suffices to take O to be a trace or products of traces of the $\Pi \equiv \sigma^A \pi_A$ and the $\bar{\Psi} \equiv \sigma^A \bar{\psi}_A$. Examples of such operators are

$$O_1 = Q = Tr\left(\bar{\Psi}\Pi\right) \ , \qquad O_2 = Tr\left(\bar{\Psi}\bar{\Psi}\Pi\right) \ ,$$

$$O_3 = Tr\left(\bar{\Psi}\bar{\Psi}\bar{\Psi}\right) = \frac{1}{2} f_{ABC}\bar{\psi}_A\bar{\psi}_B\bar{\psi}_C \quad , \tag{25}$$

$O' = Tr\left(\bar{\Psi}\Pi\Pi\right) Tr\left(\bar{\Psi}\bar{\Psi}\Pi\right)$, etc.. When these operators act on the states $|\,k\rangle_0$ of Eq.(17), energy eigenstates are produced, although they might not be new states. For example, $Tr\left(\Pi\Pi\right)$ produces the same eigenstate up to a factor of $2E_s$. Whether a new eigenstate arises also depends on the group G: When $G = SU(2)$, $Tr\left(\Pi\Pi\Pi\right)$ gives zero because the symmetric "d symbol" vanishes; for $G = SU(n)$ with $n \geq 3$, $Tr\left(\Pi\Pi\Pi\right)$ yields a new state. Like the $|\,k\rangle_0$, the O-generated states are not normalizable because the spectrum is continuous.[1] Although an infinite number of O can be constructed, only a finite number generate independent states. It may be that all eigenstates of H can be obtained by applying the O to the $|\,k\rangle_0$.

Let us verify this conjecture for $G = SU(2)$. In doing so, we shall also illustrate the reduction procedure of Sect.II for going from $U(n)$ to $SU(n)$. Factorizing the wave function in Eq.(17) for the $n = 2$ case yields

$$\frac{1}{\lambda_1 - \lambda_2}\left(\exp\left[ik_1\lambda_1 + ik_2\lambda_2\right] - \exp\left[ik_2\lambda_1 + ik_1\lambda_2\right]\right) =$$

$$\exp\left[\frac{i}{2}\left(k_1 + k_2\right)\left(\lambda_1 + \lambda_2\right)\right] \times$$

$$\frac{1}{\lambda_1 - \lambda_2}\left(\exp\left[\frac{i}{2}\left(k_1 - k_2\right)\left(\lambda_1 - \lambda_2\right)\right] - \exp\left[-\frac{i}{2}\left(k_1 - k_2\right)\left(\lambda_1 - \lambda_2\right)\right]\right)$$

$$= i\sqrt{2}\bar{k}\exp\left[\frac{i}{2}\left(k_1 + k_2\right)\left(\lambda_1 + \lambda_2\right)\right]\frac{1}{\bar{k}r}\sin\left(\bar{k}r\right) \quad , \tag{26}$$

where

$$\bar{k} \equiv \frac{k_1 - k_2}{\sqrt{2}} \quad , \tag{27}$$

[1] Certain states might be badly non-normalizable. The issue of which states should be retained in the Hilbert space goes beyond the scope of the present work.

and

$$r \equiv \sqrt{\phi_A \phi_A} \Rightarrow r = \frac{|\lambda_1 - \lambda_2|}{\sqrt{2}} \quad , \tag{28}$$

which follows from $\Phi = \sigma_A \phi_A$ after diagonalization: $\Phi \to \phi_3 = (\lambda_1 - \lambda_2)/\sqrt{2}$ with $\phi_1 = \phi_2 = 0$. Letting $\mathcal{N}^{-1} = i\sqrt{2}\bar{k}$, one obtains the wave function in $\mid k\rangle_0$ in Eq.(9) since the factorized form in Eq.(26) leads to

$$f_{U(1)} = \exp\left[i\bar{k}_0\bar{\lambda}_0\right] \ , \qquad f_{SU(2)} = \frac{1}{\bar{k}r}\sin\left(\bar{k}r\right) \quad , \tag{29}$$

Here, $\bar{k}_0 = (k_1 + k_2)/\sqrt{2}$ and $\bar{\lambda}_0 = (\lambda_1 + \lambda_2)/\sqrt{2}$. The energy separates as in Eq.(23) with

$$E_{U(1)} = \frac{1}{2}\bar{k}_0^2 \ , \qquad E_{SU(2)} = \frac{1}{2}\bar{k}^2 \quad . \tag{30}$$

Finally, a short calculation shows that when the O_i in Eq.(25) are applied to $\mid k\rangle_0 = f_{SU(2)}\mid +\rangle$, the states $\mid k\rangle_i$ in Eq.(9) are generated up to an overall normalization.

IV. Perturbative Solutions for $N > 2$

When more than two supersymmetries are present, the matrix models no longer appear to be exactly solvable. For $N > 2$, the degrees of freedom are a real gauge potential A_B, a set of real scalar ϕ_B^m, and a set of fermions ψ_B^α, where m and α label the different sets. Quantization leads to hamiltonians of the form[27]

$$H = \frac{1}{2}\pi_A^m\pi_A^m + \frac{1}{4}g^2\left(f_{ABC}\phi_B^l\phi_C^m\right)\left(f_{ADE}\phi_D^l\phi_E^m\right) + igf_{ABC}\bar{\psi}_A^\alpha\phi_B^m\Gamma_{\alpha\beta}^m\psi_C^\beta \quad , \tag{31}$$

the Gauss constraints

$$G_A = f_{ABC}\left(\phi_B^m\pi_C^m - i\bar{\psi}_B^\alpha\psi_C^\alpha\right) \quad , \tag{32}$$

and the commutation relations $\left[\phi_A^l, \pi_B^m\right] = i\delta^{lm}\delta_{AB}$ and $\left\{\psi_A^\alpha, \bar{\psi}_B^\beta\right\} = \delta^{\alpha\beta}\delta_{AB}$. In Eq.(31), the $\Gamma_{\alpha\beta}^m$, for $m = 1, 2, \ldots, p$, are matrix representations of an $SO(p)$ Clifford algebra. For example, when $N = 4$, $p = 3$, $\alpha = 1$ or 2, and the $\Gamma_{\alpha\beta}^m$ are 2×2 Pauli matrices. When $N = 16$, $p = 9$, $\alpha = 1, 2, \ldots, 16$, the $\Gamma_{\alpha\beta}^m$ are 16×16 real matrices satisfying $\left[\Gamma^l, \Gamma^m\right] = 2\delta^{lm}$, and the fermions are real. It is possible to organize the 16 real fermions into 8 complex ones at the cost of making less manifest group properties.

Because the potential-energy terms in Eq.(31) are no longer proportional to Gauss law constraints, the full effect of the interactions is felt so that the equation $H\mid s\rangle =$

$E_s|\, s\rangle$ is difficult to solve. A perturbative approach is possible. To begin a perturbative expansion, solutions to the $g = 0$ system must be known. Such solutions are obtainable using the methods of Sections II and III because, when g is zero, the hamiltonian is a sum of p independent $N = 2$ hamiltonians. Hence, the system factorizes. The analog of the state in Eq.(17) is

$$|\, k\rangle_0 = \mathcal{N} \prod_m \left[\mathcal{M}_m^{-1} \sum_{\sigma \in S_n} (-1)^\sigma \exp\left(i \sum_j k_{\sigma(j)}^{(m)} \lambda_j^{(m)} \right) \right] |\, +\rangle \quad , \tag{33}$$

where $|\, +\rangle$ is annihilated by all the ψ_A^α, $k_j^{(m)}$ are momenta for the mth sector, and

$$\mathcal{M}_m^{-1} \equiv \prod_{i<j} \left(\lambda_i^{(m)} - \lambda_j^{(m)} \right)^{-1} \quad , \tag{34}$$

where the $\lambda_j^{(m)}$ are the eigenvalues of the matrix $\phi^m = \sigma^A \phi_A^m$. The energy is

$$E = \frac{1}{2} \sum_{j,m} \left(k_j^{(m)} \right)^2 \quad . \tag{35}$$

Additional eigenstates are generated by applying to $|\, k\rangle_0$ operators O that are gauge invariant and that are functionals only of the π_B^m and fermions. Such O involve a trace or products of traces of the $\Pi^m \equiv \sigma^A \pi_A^m$ and the $\bar{\Psi}^\alpha \equiv \sigma^A \bar{\psi}_A^\alpha$. Examples of such operators are $Tr\left(\Pi^l \Pi^m\right)$, $Tr\left(\bar{\Psi}^\alpha \bar{\Psi}^\beta\right)$ for $\alpha \neq \beta$, $Tr\left(\bar{\Psi}^\alpha \Pi^m\right) Tr\left(\bar{\Psi}^\alpha \bar{\Psi}^\beta \Pi^l\right)$, etc..

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References

[1] M. Duff, P. Howe, T. Inami and K. Stelle, Phys. Lett. **191B** (1987) 70.

[2] E. Witten, Nucl. Phys. **B443** (1995) 85; hep-th/9503124.

[3] P. K. Townsend, Phys. Lett. **350B** (1995) 184; hep-th/9501068.

[4] M. J. Duff, to appear in Intl. J. Mod. Phys. **A**. and references therein; hep-th/9608177.

[5] P. K. Townsend, *Four Lectures on M-Theory*, and references therein; hep-th/9612121.

[6] J. H. Schwartz, *Lectures on Superstring and M-theory Dualities*, and references therein; hep-th/9607201.

[7] C. Montonen and D. Olive, Phys. Lett. **72B** (1977) 117.

[8] D. Olive and E. Witten, Phys. Lett. **83B** (1979) 321.

[9] V. P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. **B287** (1987) 402.

[10] M. Dine, P. Huet, and N. Seiberg, Nucl. Phys. **B322** (1989) 301.

[11] J. Dai, R. G. Leigh and J. Polchinski, Intl. Mod. Phys. Lett. **A4** (1989) 2073.

[12] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; hep-th/9407087; Nucl. Phys. **B431** (1994) 484; hep-th/9408099.

[13] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438** (1995) 109; hep-th/9410167.

[14] A. Sen, *Unification of String Dualities*, and references therein; hep-th/9609176.

[15] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **244** (1994) 77, and references therein; hep-th/9401139.

[16] M. J. Duff, Intl. Mod. Phys. Lett. **A11** (1996) 4031, and references therein.

[17] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724; hep-th/9510017.

[18] J. Polchinski, *TASI lectures on D-branes*, and references therein; hep-th/9611050.

[19] J. Hughes, J. Liu and J. Polchinski, Phys. Lett. **180B** (1986) 370.

[20] P. K. Townsend, *Recent Problems in Mathematical Physics*, and references therein, Proceedings of the 13th GIFT Seminar on Theoretical Physics: Salamanca, Spain (June, 1992).

[21] M. J. Duff, *Supermembranes*, and references therein; hep-th/9611203.

[22] B. De Wit, J. Hoppe and H. Nicolai, Nucl. Phys. **B305 [FS 23]** (1988) 545.

[23] B. De Wit, M. Lüscher and H. Nicolai, Nucl. Phys. **B320** (1989) 135.

[24] E. Witten, Nucl. Phys. **B460** (1995) 335; hep-th/9510135.

[25] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M Theory as a Matrix Model: A Conjecture*; hep-th/9610043.

[26] E. Witten, Nucl. Phys. **B185** (1981) 513.

[27] M. Claudson and M. B. Halpern, Nucl. Phys. **B250** (1985) 689.

[28] V. A. Kostelecký, O. Lechtenfeld, W. Lerche, S. Watamura and S. Samuel, Nucl. Phys. **B288** (1987) 173.